Lecture 23: Composition of Relations, Transitive Closure and Warshall’s Algorithm

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1 Composition of Relations

In this section we will study what is meant by composition of relations and how it can be obtained.

Suppose that we have three sets $A$, $B$ and $C$; a relation $R$ defined from $A$ to $B$, and a relation $S$ defined from $B$ to $C$. We can now define a new relation known as the composition of $R$ and $S$, written as $S \circ R$. This new relation is defined as follows. If $a$ is an element in $A$ and $c$ is an element in $C$, then $a(S \circ R)c$ if and only if there exists some element $b$ in $B$, such that $aRb$ and $bSc$. This means that we have a relation $S \circ R$ from $a$ to $c$, if and only if we can reach from $a$ to $c$ in two steps; i.e. from $a$ to $b$ related by $R$ and from $b$ to $c$ related by $S$. In this manner relation $S \circ R$ can be interpreted as $R$ followed by $S$, since this is the order in which the two relations need to be considered, first $R$ then $S$.

† Example 1:

Let us try to understand this better through an example. Let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 1), (1, 3), (2, 4), (3, 2)\}$, and $S = \{(1, 4), (1, 3), (2, 3), (3, 1), (4, 1)\}$. Find $S \circ R$.

Solution:

Here we see that $(1, 2) \in R$ and $(2, 3) \in S$. This gives us $(1, 3) \in S \circ R$. Similarly we can proceed with the others:

- $(1, 1) \in R$ and $(1, 4) \in S$ \Rightarrow $(1, 4) \in S \circ R$
- $(1, 1) \in R$ and $(1, 3) \in S$ \Rightarrow $(1, 3) \in S \circ R$
- $(1, 3) \in R$ and $(3, 1) \in S$ \Rightarrow $(1, 1) \in S \circ R$
- $(2, 4) \in R$ and $(4, 1) \in S$ \Rightarrow $(2, 1) \in S \circ R$
- $(3, 2) \in R$ and $(2, 3) \in S$ \Rightarrow $(3, 3) \in S \circ R$

\Rightarrow $S \circ R = \{(1, 3), (1, 4), (1, 1), (2, 1), (3, 3)\}$.

§ Theorem 1

Let $R$ be a relation from $A$ to $B$ and let $S$ be a relation from $B$ to $C$. Then, if $A_1$ is any subset of $A$, we have:

$(S \circ R)(A_1) = S(R(A_1))$
1 COMPOSITION OF RELATIONS

Proof
If an element \( z \in C \) is in \((S \circ R)(A_1)\), then \( x(R \circ S)z \) for some \( x \in A_1 \). By the definition of composition, this means that \( xRy \) and \( ySz \) for some \( y \in B \).

So now we have \( z \in S(y) \) and \( y \in R(x) \Rightarrow z \in S(R(x)) \).

Since \( \{x\} \subseteq A_1 \), we can also say that \( S(R(x)) \subseteq S(R(A_1)) \). Now, since \( z \in S(R(x)) \), therefore \( z \in S(R(A_1)) \) also. This means that \((S \circ R)(A_1) \subseteq S(R(A_1))\).

Conversely, suppose \( z \in S(R(A_1)) \). Then \( z \in S(y) \) for some \( y \in R(A_1) \). Similarly, \( y \in R(x) \) for some \( x \in A_1 \).

This means that \( xRy \) and \( ySz \). So from the definition of composition we can say \( x(S \circ R)z \). Thus \( z \in (S \circ R)(x) \). Since \( \{x\} \subseteq A_1 \), we can say that \( (S \circ R)(x) \subseteq (S \circ R)(A_1) \).

Hence \( z \) also belongs to \( (S \circ R)(A_1) \). So \( S \circ R \subseteq S(R(A_1)) \).

This proves the theorem.

† Example 2:
Let \( A = \{a, b, c\} \) and let \( R \) and \( S \) be relations on \( A \) whose matrices are:

\[
M_R = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]

\[
M_S = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

Find \( S \circ R \).

Solution:
We see from the matrices that:

\begin{itemize}
  \item \((a, a) \in R \) and \((a, a) \in S \Rightarrow (a, a) \in S \circ R \)
  \item \((a, c) \in R \) and \((c, a) \in S \Rightarrow (a, a) \in S \circ R \)
  \item \((a, c) \in R \) and \((c, c) \in S \Rightarrow (a, c) \in S \circ R \)
\end{itemize}

It is easily seen that \((a, b) \notin S \circ R \) since, if we had \((a, x) \in R \) and \((x, b) \in S \), then from Matrix \( M_R \) we know that \( x \) would have to be either \( a \) or \( c \); but from matrix \( M_S \) we know that neither \( (a, b) \) nor \( (c, b) \) is an element of \( S \).

Hence we see that the first row of \( M_{S \circ R} \) is \( 1 0 1 \).

Proceeding in a similar manner we get:

\begin{itemize}
  \item \((b, a) \in R \) and \((a, a) \in S \Rightarrow (b, a) \in S \circ R \)
\end{itemize}
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- \((b, b) \in R \) and \((b, b) \in S \) \(\Rightarrow\) \((b, b) \in S \circ R\)
- \((b, b) \in R \) and \((b, c) \in S \) \(\Rightarrow\) \((b, c) \in S \circ R\)
- \((b, c) \in R \) and \((c, a) \in S \) \(\Rightarrow\) \((b, a) \in S \circ R\)
- \((b, c) \in R \) and \((c, c) \in S \) \(\Rightarrow\) \((b, c) \in S \circ R\)

Hence the second row of \(M_{S \circ R}\) is \(1\ 1\ 1\).

- \((c, b) \in R \) and \((b, b) \in S \) \(\Rightarrow\) \((c, b) \in S \circ R\)
- \((c, b) \in R \) and \((b, c) \in S \) \(\Rightarrow\) \((c, c) \in S \circ R\)

Hence the third row of \(M_{S \circ R}\) is \(0\ 1\ 1\).

Therefore the composition matrix is

\[
M_{S \circ R} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\]

Now we shall deduce an important and useful result.

Let us consider three sets, \(A = \{a_1, \ldots, a_n\}\), \(B = \{b_1, \ldots, b_p\}\) and \(C = \{c_1, \ldots, c_m\}\); and relation \(R\) defined from \(A\) to \(B\), and \(S\) defined from \(B\) to \(C\). The Boolean matrices \(M_R\) and \(M_S\) are of sizes \(n \times p\) and \(p \times m\) respectively. Let us represent \(M_R\) as \([r_{ij}]\), \(M_S\) as \([s_{ij}]\) and \(M_{S \circ R}\) as \([t_{ij}]\). Now \(t_{ij} = 1 \iff (a_i, c_j) \in S \circ R\), which means that for some \(k\) between 1 and \(p\), \((a_i, b_k) \in R\) and \((b_k, c_j) \in S\), that is, \(r_{ik} = 1\) and \(s_{kj} = 1\). In other words, if \(r_{ik} = 1\) and \(s_{kj} = 1\) then \(t_{ij} \leftarrow 1\). This is identical to the condition needed for \(M_R \odot M_S\) to have a 1 in position \((i,j)\).

Hence we can say that:

\[
M_{S \circ R} = M_R \odot M_S.
\]

In the special case when we have \(S = R\), \(S \circ R = R^2\) and \(M_{S \circ R} = M_{R^2} = M_R \odot M_R\).

§ Theorem 2

Let \(A\), \(B\), \(C\) and \(D\) be sets, \(R\) a relation from \(A\) to \(B\), \(S\) a relation from \(B\) to \(C\) and \(T\) a relation from \(C\) to \(D\). Then

\[
T \circ (S \circ R) = (T \circ S) \circ R
\]

Proof

Let the Boolean matrices for the relations \(R\), \(S\) and \(T\) be \(M_R\), \(M_S\) and \(M_T\) respectively. As was shown in Example 2, the Boolean matrix product represents the matrix of composition, i.e. \(M_{S \circ R} = M_R \odot M_S\).

Thus we have:
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\[ M_{T \circ (S \circ R)} = M_{S \circ R} \odot M_T = (M_R \odot M_S) \odot M_T \]

Similarly we have:

\[ M_{(T \circ S) \circ R} = M_R \odot M_{T \circ S} = M_R \odot (M_S \odot M_T) \]

Now, we know that Boolean multiplication is associative. This implies:

\[ (M_R \odot M_S) \odot M_T = M_R \odot (M_S \odot M_T) \]

\[ \Rightarrow M_{T \circ (S \circ R)} = M_{(T \circ S) \circ R} \]

Now since the Boolean matrices for these relations are the same,

\[ \Rightarrow T \circ (S \circ R) = (T \circ S) \circ R \]

This completes the proof.

**Note:**

In general \( R \circ S \neq S \circ R \). This can be shown by the following example.

Let \( A = \{a, b\} \), \( R = \{(a, a), (b, a), (b, b)\} \) and \( S = \{(a, b), (b, a), (b, b)\} \). In this case \( S \circ R = \{(a, b), (b, a), (b, b)\} \), while \( R \circ S = \{(a, a), (a, b), (b, a), (b, b)\} \). Here, we can clearly see that \( R \circ S \neq S \circ R \).

§ **Theorem 3**

Let \( A, B \) and \( C \) be three sets, \( R \) a relation from \( A \) to \( B \), and \( S \) a relation from \( B \) to \( C \). Then \((S \circ R)^{-1} = R^{-1} \circ S^{-1}\).

**Proof**

Let \( c \in C \) and \( a \in A \). Then \((c, a) \in (S \circ R)^{-1}\) if and only if \((a, c) \in S \circ R\). This is true if and only if there is a \( b \in B \) with \((a, b) \in R\) and \((b, c) \in S\). Finally, this is equivalent to the statement that \((c, b) \in S^{-1}\) and \((b, a) \in R^{-1}\); that is, \((c, a) \in R^{-1} \circ S^{-1}\) (by the definition of composition). This proves that

\[ (S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1} \]

Conversely, let \((c, a) \in R^{-1} \circ S^{-1}\), then for some \( b \in B \), \((c, b) \in S^{-1}\) and \((b, a) \in R^{-1}\) (again by the definition of composition). This implies that \((a, b) \in R\) and \((b, c) \in S\). Hence we can say that \((a, c) \in S \circ R\), which also means that \((c, a) \in (S \circ R)^{-1}\). This proves that

\[ R^{-1} \circ S^{-1} \subseteq (S \circ R)^{-1} \]

Since \((S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}\) and \( R^{-1} \circ S^{-1} \subseteq (S \circ R)^{-1}\), we can say that:

\[ (S \circ R)^{-1} = R^{-1} \circ S^{-1} \]

This completes the proof.
2 Transitive Closure

A relation \( R \) is said to be transitive if for every \((a, b) \in R\) and \((b, c) \in R\) there is a \((a, c) \in R\).

A transitive closure of a relation \( R \) is the smallest transitive relation containing \( R \).

Suppose that \( R \) is a relation defined on a set \( A \) and that \( R \) is not transitive. Then the transitive closure of \( R \) is the connectivity relation \( R^\infty \). We will now try to prove this claim.

§ Theorem 4

Let \( R \) be a relation on a set \( A \). Then \( R^\infty \) is the transitive closure of \( R \).

Proof

We need to prove that \( R^\infty \) is transitive and also that it is the smallest transitive relation containing \( R \).

If \( a \) and \( b \in A \), then \( aR^\infty b \) if and only if there exists a path in \( R \) from \( a \) to \( b \). If \( aR^\infty b \) and \( bR^\infty c \), then we can say that \( aR^\infty c \). This is because \( aR^\infty b \) means that there exists a path from \( a \) to \( b \) in \( R \), similarly \( bR^\infty c \) means that there exists a path from \( b \) to \( c \) in \( R \). Hence there will also exist a path from \( a \) to \( c \) in \( R \). (We can simply start along the path from \( a \) to \( b \) and then continue along the path from \( b \) to \( c \). This will give us the path from \( a \) to \( c \).) This proves that \( R^\infty \) is transitive.

Now let us consider a transitive relation \( S \) (containing \( R \)) i.e. \( R \subseteq S \). Since \( S \) is transitive we can say that \( S^n \subseteq S \forall n \). (This means that if there is a path of length \( n \) from \( a \) to \( b \), then \( aSb \), which is true as \( S \) is a transitive relation.) Now, \( S^\infty = \bigcup_{n=1}^\infty S^n \). Hence \( S^\infty \subseteq S \). Since \( R \subseteq S \), therefore \( R^\infty \subseteq S^\infty \), and as \( S^\infty \subseteq S \), we can say that \( R^\infty \subseteq S \). This means that \( R^\infty \) is the smallest of all transitive relations on \( A \) that contain \( R \).

As \( R^\infty \) satisfies both the properties, we can say that \( R^\infty \) is the transitive closure of \( R \) on set \( A \). This completes our proof.

\( \checkmark \) Note:
If we include the identity relation \( \Delta \) then \( R^\infty \cup \Delta \) is the reachability relation \( R^* \).

† Example 3:

Let \( A = \{1, 2, 3, 4\} \), and let \( R = \{(1, 2), (2, 3), (3, 4), (2, 1)\} \). Find the transitive closure of \( R \).

Solution:
Method 1: Using Digraph
We can determine $R^\infty$ by geometrically computing all paths from the digraph. From vertex 1 we have paths to vertices 1, 2, 3 and 4. So the ordered pairs $(1, 1), (1, 2), (1, 3)$ and $(1, 4) \in R^\infty$. From vertex 2 we have paths to vertices 1, 2, 3 and 4. This gives us the ordered pairs $(2, 1), (2, 2), (2, 3)$ and $(2, 4)$. From vertex 3 we have only one path to vertex 4. This gives us the ordered pair $(3, 4)$. From vertex 4 we do not have any paths. So we have

$$R^\infty = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$$

**Method 2: Using Matrices**

Writing down the Boolean matrix we get

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
If we compute the higher powers of $M_R$ we get:

\[
(M_R)^2 = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
(M_R)^3 = \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
(M_R)^4 = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Thus we observe that $(M_R)^n$ equals $(M_R)^2$ if $n$ is even, and equals $(M_R)^3$ if $n$ is odd and greater than 1. Hence we get,

\[
M_R^\infty = M_R \lor (M_R)^2 \lor (M_R)^3
\]

Thus,

\[
M_R^\infty = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

which is nothing but the matrix representation of the relation we obtained earlier by the digraph method (1). Thus we see that we need not consider all the powers of $R^n$ to obtain $R^\infty$ when the set $A$ is finite. This result always holds good, as we will now prove.
\section*{Theorem 5}

Let $A$ be a set with $|A| = n$, and let $R$ be a relation on $A$. Then
\[ R^\infty = R \cup R^2 \cup \ldots \cup R^n \]
i.e. powers of $R$ greater than $n$ need not be considered to compute $R^\infty$.

\textbf{Proof}

Let $a$ and $b \in A$, and let $a, x_1, x_2, \ldots, x_m, b$ be a path from $a$ to $b$ in $R$; i.e. $(a, x_1), (x_1, x_2), \ldots, (x_m, b) \in R$. Now if $x_i$ and $x_j$ correspond to the same vertex for some $i < j$, then the path from $a$ to $b$ can be distinctly divided into three regions. First, a path from $a$ to $x_i$; second, a path from $x_i$ to $x_j$; and lastly, a path from $x_j$ to $b$. Here we see that the second path forms a closed loop as $x_i = x_j$. So we can eliminate it altogether and put the first and third paths together to give us a shorter path. In a similar manner we can keep eliminating all the loops we get later on in the path too, to give us a path $a, x_1, x_2, \ldots, x_k, b$ where all of $x_1, x_2, \ldots, x_k$ are distinct. This path is the shortest one possible from $a$ to $b$. 

Figure 2: Shows how a closed loop can be eliminated to give the shortest path. Here, \( i = 2 \) and \( j = 5 \).

Now let us consider the case when \( a \neq b \). Since the total number of elements in the set \( A \) is \( n \), the maximum path length we can get is \( n - 1 \). If \( a = b \), then we get the maximum path length as \( n \) (as \( |A| = n \) and all the vertices except \( a \) and \( b \) are distinct). 'There is a path from \( a \) to \( b \) in \( R \)' is equivalent to \( aR^\infty b \). And if \( aR^\infty b \) (i.e. there is a path from \( a \) to \( b \) in \( R \)), from the preceding discussion we know that \( aR^k b \) for some \( k, 1 \leq k \leq n \) (as the maximum path length possible is \( n \)). Thus \( R^\infty = R \cup R^2 \cup \ldots \cup R^n \). Hence the theorem is proved.


3 Warshall’s Algorithm

Need for Warshall’s Algorithm

The methods used to solve Example 3 have certain drawbacks. The graphical method is un-systematic and impractical for large sets. The matrix method is better than the graphical method and can be implemented with the help of a program, but it tends to become costly in terms of time and space requirement in the case of large matrices. So this method is inefficient too. The Warshall’s algorithm as described next helps to overcome these drawbacks.

The Procedure

Let $A$ be a set such that $A = \{a_1, a_2, \ldots, a_n\}$ and $R$ be a relation defined on $A$. Considering a general path $x_1, x_2, \ldots, x_m$ in $R$, all vertices except the two at the extremes, i.e. $x_1$ and $x_m$, are called the interior vertices. Now, we define a Boolean matrix $W_k$ ($1 \leq k \leq n$) as follows. $W_k$ has a $1$ in position $(i, j)$ if and only if there is a path from $a_i$ to $a_j$ in $R$ whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots, a_k\}$.

If we take $k = n$, any vertex must come from the set $\{a_1, a_2, \ldots, a_n\}$, and hence $W_n$ has a $1$ in position $(i, j)$ if and only if some path in $R$ connects $a_i$ with $a_j$. This means that $W_n = R^\infty$. If we take $W_0$ equal to $M_R$, we will get a sequence $W_0, W_1, W_2, \ldots, W_n$, where $W_0$ corresponds to $M_R$ and $W_n$ corresponds to $M_R^\infty$. The Warshall’s algorithm gives us a way to compute each matrix $W_k$ from the previous matrix $W_{k-1}$. Hence we begin with the matrix representation of $R$ and proceed to $R^\infty$. The computation of $W_k$ is different from that of the powers of $M_R$ and saves time considerably as the steps involved are fewer and less complicated.

Let us suppose $W_k = [p_{ij}]$ and $W_{k-1} = [s_{ij}]$. If $p_{ij} = 1$, then there exits a path from $a_i$ to $a_j$ in $R$ whose interior vertices are from $\{a_1, a_2, \ldots, a_k\}$. If the vertex $a_k$ is not an interior vertex in this path, then all the interior vertices are from $\{a_1, a_2, \ldots, a_{k-1}\} \Rightarrow s_{ij} = 1$. If the vertex $a_k$ is an interior vertex in the path, then there will be a subpath from $a_i$ to $a_k$ and another subpath from $a_k$ to $a_j$. All the interior vertices in the path from $a_i$ to $a_j$ are distinct (Theorem 5). So $a_k$ appears in the path only once, and hence all the interior vertices in the two subpaths mentioned earlier come from the set $\{a_1, a_2, \ldots, a_{k-1}\}$. This means that $s_{ik} = 1$ and $s_{kj} = 1$.

Thus $p_{ij} = 1$ only under two conditions. If and only if either $s_{ij} = 1$, OR $s_{ik} = 1$ and $s_{kj} = 1$. So if $W_{k-1}$ has a $1$ in the position $(i, j)$, so will $W_k$. A new $1$ can be added at position $(i, j)$ in $W_k$ if and only if column $k$ in $W_{k-1}$ has a $1$ at position $i$ and row $k$ of $W_{k-1}$ has a $1$ at position $j$. So the procedure for calculating $W_k$ from $W_{k-1}$ is:

1. Transfer to $W_k$ all the 1’s in $W_{k-1}$.

2. List the locations $y_1, y_2, \ldots$, in column $k$ of $W_{k-1}$ where the entry is 1, and the locations $z_1, z_2, \ldots$, in row $k$ of $W_{k-1}$ where the entry is 1.
3. Put 1’s at all positions \((y_i, z_j)\) of \(W_k\) (if not already present).

† Example 4:

Find the transitive closure of \(R\) defined in Example 3.

Solution:

\[
W_0 = M_R = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Here we have \(n = 4\).

To find \(W_1, k = 1\). We can see that \(W_0\) has 1’s in column 1 at location 2, and in row 1 at location 2. Thus \(W_1\) has a new 1 at position \((2, 2)\).

\[
W_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

For \(W_2, k = 2\). \(W_1\) has 1’s in column 2 at locations 1 and 2, and in row 2 at locations 1, 2 and 3. So the new 1’s would go to positions \((1, 1), (1, 2), (1, 3), (2, 1), (2, 2)\) and \((2, 3)\) (if not already there).

\[
W_2 = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

For \(W_3, k = 3\). \(W_2\) has 1’s in column 3 at locations 1 and 2, and in row 3 at location 4. So the new 1’s would come at positions \((1, 4)\) and \((2, 4)\) (if not already there).

\[
W_3 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

For \(W_4, k = 4\). \(W_3\) has 1’s in column 4 at locations 1, 2 and 3 but no 1’s in row 4. So no new 1’s are added. Hence \(W_4 = W_3\). This gives us the matrix representation of \(R^\infty\) which is the same as that obtained in Example 3.
The Algorithm

To find the matrix CLOSURE of the transitive closure of a relation \( R \) whose \( n \times n \) matrix representation is \( \text{MAT} \).

**WARSHALL**

1. \( \text{CLOSURE} \leftarrow \text{MAT} \)
2. for \( k = 1 \) to \( n \)
3. for \( i = 1 \) to \( n \)
4. for \( j = 1 \) to \( n \)
5. \( \text{CLOSURE}(i, j) \leftarrow \text{CLOSURE}(i, j) \lor (\text{CLOSURE}(i, k) \land \text{CLOSURE}(k, j)) \)

This algorithm involves three for loops, with two of them nested. Each loop iterates from 1 to \( n \). This gives us a time complexity of \( O(n^3) \). If we were to find the transitive closure using the matrix multiplication method we would get a time complexity of \( O(n^4) \). Each time a matrix multiplication is performed the time complexity is \( O(n^3) \) as there are three loops running (two nested) from 1 to \( n \). The matrix multiplications are carried out a total of \( n - 1 \) times to find matrices \( (M_R)^2, (M_R)^3, \ldots, (M_R)^n \), since \( M_R^\infty = M_R \lor (M_R)^2 \lor \ldots \lor (M_R)^n \). So the number of steps involved are \( n^3(n - 1) \), giving us a time complexity of \( O(n^4) \). Thus we see that Warshall’s algorithm is surely simpler and more efficient than the matrix multiplication method.

An Application of Warshall’s Algorithm

§ Theorem 6

If \( R \) and \( S \) are equivalence relations on a set \( A \), then the smallest equivalence relation containing both \( R \) and \( S \) is \( (R \cup S)^\infty \).

Proof

We know that a relation is reflexive if and only if it contains the identity or equality relation \( \Delta \). Since both \( R \) and \( S \) are reflexive, \( \Delta \subseteq R \) and \( \Delta \subseteq S \). This implies that \( \Delta \subseteq R \cup S \subseteq (R \cup S)^\infty \). So \( (R \cup S)^\infty \) is also reflexive.

\( R \) and \( S \) are symmetric. Let us consider \( (a, b) \in R \Rightarrow (b, a) \in R \) (as \( R \) is symmetric). Since \( R \subseteq (R \cup S) \subseteq (R \cup S)^\infty \), therefore \( (a, b) \) and \( (b, a) \in (R \cup S)^\infty \). This implies that \( (R \cup S)^\infty \) is also symmetric. This can be proved in a similar manner by taking \( (a, b) \in S \) instead of \( R \).

The property of transitive closure tells us that any relation \( T^\infty \) is the smallest transitive relation containing the relation \( T \). Applying this property to \( R \cup S \), we can conclude that \( (R \cup S)^\infty \) is the smallest transitive relation containing \( (R \cup S) \). Since it is transitive
also, it is an equivalence; and since it contains \((R \cup S)\), it contains both \(R\) and \(S\). Hence we have proved that \((R \cup S)\) is the smallest equivalence relation containing both \(R\) and \(S\).

† Example 5:

Let \(A = \{1, 2, 3, 4, 5\}\), \(R = \{(1,1), (1, 2), (2, 1), (2, 2), (3,3), (3,4), (4,3), (4,4), (5,5)\}\), and \(S = \{(1,1), (2,2), (3,3), (4,4), (5,4), (5,5)\}\) (where \(R\) and \(S\) are equivalence relations). The partition \(A/R\) of \(A\) corresponding to \(R\) is \({\{1,2\}, \{3,4\}, \{5\}\}\), and the partition \(A/S\) of \(A\) corresponding to \(S\) is \({\{1\}, \{2\}, \{3\}, \{4,5\}\}\). Find the smallest equivalence relation containing \(R\) and \(S\), and compute the partition of \(A\) that it produces.

Solution:

\[
M_R = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

and

\[
M_S = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

So,

\[
M_{R \cup S} = M_R \vee M_S = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

We now compute \(M_{(R \cup S)^\infty}\) using Warshall’s algorithm. First \(W_0 = M_{R \cup S}\). Next we compute \(W_1\) \((k = 1)\). \(W_0\) has 1’s at locations 1 and 2 in column 1, and at locations 1 and 2 in row 1. No new 1’s need to be added as 1’s are already present at positions (1, 1), (1, 2), (2, 1) and (2, 2). So \(W_1 = W_0\).

We now compute \(W_2\) \((k = 2)\). Since \(W_1\) has 1’s at locations 1 and 2 of column 2, and at locations 1 and 2 of row 2, no new 1’s need to be added. So \(W_2 = W_1\).

Next we move on to \(W_3\) \((k = 3)\). Since \(W_2\) has 1’s at locations 3 and 4 of column 3, and at locations 3 and 4 of row 3, no new 1’s need to be added. So \(W_3 = W_2\).
Now we compute $W_4 \ (k = 4)$. $W_3$ has 1’s at locations 3, 4 and 5 of column 4, and at locations 3, 4 and 5 of row 4. So new 1’s need to be added at positions (3, 5) and (5, 3) of $W_3$ to get $W_4$. Thus

$$W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

To compute $W_5 \ (k = 5)$, we see that, since $W_4$ has 1’s at locations 3, 4 and 5 of column 5, and at locations 3, 4 and 5 of row 5, no new 1’s need to be added. So $W_5 = W_4$.

$\therefore \ (R \cup S)^\infty = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$.

The corresponding partition is $\{\{1, 2\}, \{3, 4, 5\}\}$. 