Foundations of Computing Science (CS60001)
Tutorial - 06

14^{th}-15^{th} November, 2011

Lessons Covered : Time Complexity

1. Let

$$MODEXP = \{\langle\ a, b, c, p \mid a, b, c \text{ and } p \text{ are binary integers such that } a^b \equiv c \pmod p\}.$$

   Show that $MODEXP \in P$.

2. Show that P is closed under $Union$, $Concatenation$, $Complement$ and $Star$ operation.

3. Show that NP is closed under $Union$, $Star$ and $Concatenation$.

4. Show that if $P = NP$, then every language $A \in P$, except $A = \emptyset$ and $A = \Sigma^*$, is $NP-Complete$.

5. Let $CNF_k = \{\langle\phi\rangle \mid \phi$ is a satisfiable $cnf - formula$ where each variable appears in at most k places$\}$.

   (a) Show that $CNF_2 \in P$.

   (b) Show that $CNF_3 \in NP - Complete$.

6. A **2cnf − formula** is and AND of clauses, where each clause is an OR of at most two literals. Let $2SAT = \{\langle\phi\rangle \mid \phi$ is a satisfiable 2cnf-formula$\}$. Show that $2SAT \in P$.

7. A **coloring** of a graph is an assignment of colors to its nodes so that no two adjacent nodes are assigned the same color. Let

$$3COLOR = \{\langle G\rangle \mid \text{the nodes of } G \text{ can be colored with three colors such that no two nodes joined by an edge have the same color}\}.$$

   Show that $3COLOR$ is $NP - Complete$. ($Hint$: Use reduction from $3 - CNF - SAT$).

8. Let $SET - SPLITTING = \{\langle S,C\rangle \mid S$ is an finite set and $C = \{C_1, C_2, \ldots, C_k\}$ is a collection of subsets of S, for some $k > 0$, such that elements of S can be colored *red* or *blue* so that no C_i has all its elements colored with the same color.$\}$. Show that $SET - SPLITTING$ is $NP - Complete$.

9. Let $DOUBLE-SAT = \{\langle\phi\rangle \mid \phi$ has at least two satisfying assignments$\}$. Shoe that $DOUBLE-SAT$ is $NP - Complete$.

10. The **Subgraph − Isomorphism problem** takes two graphs G_1 and G_2 and asks whether G_1 is isomorphic to a *subgraph* of G_2. Show that **Subgraph − Isomorphism** is $NP - Complete$.

11. The $Set - Partiton\ Problem$ takes as input a set S of numbers. The question is whether the number can be partitioned into two sets A and $\overline{A} = S$ - A such that $\Sigma_{x\in A} x = \Sigma_{x\in\overline{A}} x$. Show that the $Set - Partiton$ problem is $NP - Complete$.

12. Let G represent an undirected graph. Also let

$$SPATH = \{\langle G, a, b, k\rangle \mid G \text{ contains a simple path of length at most } k \text{ from } a \text{ to } b\},$$

and

$$LPATH = \{\langle G, a, b, k \rangle \mid G \text{ contains a simple path of length at least } k \text{ from } a \text{ to } b\},$$

a) Show that $SPATH \in P$.

b) Show that $LPATH$ is $NP - Complete$. You may assume the $NP - Completeness$ of $UHAMPATH$, the Hamiltonian path problem for undirected graphs.

13. A **cut** in an undirected graph is a seperation of the vertices V into two disjoint subsets S and T. The size of the cut is the number of the edges that have one endpoint in S and the other in T. Let

$$MAX\text{-}CUT = \{\langle G, k \rangle \mid G \text{ has a cut size } k \text{ or more}\}.$$

Show that $MAX\text{-}CUT$ is $NP\text{-}Complete$.

14. (a) Show that the problem of determining the satisfiability of boolean formulas in disjunctive normal form is polynomial-time solvable.
    (b) Show that L is complete for NP if and only if $\overline{L}$ is complete for $co - NP$.
    (c) Prove that $L \leq_P \overline{L}$ if and only if $\overline{L} \leq_P L$.
    (d) Show that $\leq_P$ relation is *transitive* realtion on languages. That is, show that if $L_1 \leq_P L_2$ and $L_2 \leq_P L_3$, then $L_1 \leq_P L_3$.
    (e) Prove that $P \subseteq co - NP$.
    (f) Show that if $P = NP$, then every language $A \in P$, except $A = \emptyset$ and $A = \Sigma^*$, is NP-*Complete*.

15. Let ϕ be a boolean formula constructed from the boolean input variables $x_1, x_2, \ldots, x_k$, negations ($\neg$), AND's ($\wedge$), OR's ($\vee$), and parantheses. The Formula ϕ is a **tautology** if it evaluates to 1 for every assignment of 1 and 0 to the input variables. Define $TAUTOLOGY$ as the language of boolean formulas that are tautologies. Show that $TAUTOLOGY \in co - NP$.

16. Let $UNARY\text{-}SSUM$ be the subset sum problem in which all numbers are represented in unary. Show that $UNARY\text{-}SSUM \in P$.

17. Consider the following scheduling problem. You are given a list of Final exams F_1, F_2,...,F_k to be scheduled, and a list of students S_1,S_2,...,S_l. Each student is taking some specified subset of this exams. You must schedule these exams into slots so that no student is required to take two exams in same slot. The problem is to determine if such a schedule exists that uses only h slots. Formulate this problem as a language and show that this language is $NP\text{-}Complete$.