Tutorial 6, Question 2i

To prove

$$\{\forall x p(x,x) \wedge \forall x \forall y \forall z (p(x,y) \wedge p(y,z) \Rightarrow p(x,z)) \wedge \forall x \forall y [p(x,y) \vee p(y,x)]\} \Rightarrow \exists y \forall x p(y,x)$$

is true for any interpretation with a finite domain.

Proof

We shall prove this by induction on the size of the domain($|D|$).
Case 1:$|D| = 1$
Let a be the only element in the domain. Then, for the antecedent to be true, $p(a,a)$ must be true. Since a is the only element, the consequent also becomes true.
Case 2:$|D| = n+1$
We assume that the statement holds for all domains of size n. From a domain of size n, we create a domain of size $n+1$ by adding a new element, say a. Let b be the element present in the domain of size n, whose existence is guaranteed by the statement.(i.e, $\forall y p(b,y)$).
Now, in the new domain, for the antecedent to be true, we have $p(a,a)$ and $p(a,b)$ or $p(b,a)$.
If $p(b,a)$, we can say $\forall y p(b,y)$, since we already know from our assumption that for each element x except a, $p(b,x)$ is true.
If $p(a,b)$, we can infer that for each element x apart from a and $p(a,x)$ is true, since for each of them, $p(b,x)$ is true according to our assumption, and we also have that $\forall x \forall y \forall z (p(x,y) \wedge p(y,z) \Rightarrow p(x,z))$. Since we also know that $p(a,a)$ is true, we can say that, $\forall y p(a,y)$ is true.
Hence in either case, there exists x such that $\forall y (p(x,y))$.
To prove that it is not valid for the domain of natural numbers, we can define p as follows:

$$\forall x \forall y (p(x,y) \Leftrightarrow x \leq y)$$