Graph Theory: Graph Coloring

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K-coloring

A k-coloring of G is a labeling $f:V(G) \rightarrow \{1,\ldots,k\}$.

- The labels are colors
- The vertices with color $i$ are a color class
- A k-coloring is proper if $x \leftrightarrow y$ implies $f(x) \neq f(y)$
- A graph G is k-colorable if it has a proper k-coloring
- The chromatic number $\chi(G)$ is the min $k$ s.t. G is k-colorable
- If $\chi(G) = k$, then G is k-chromatic
- If $\chi(G) = k$, but $\chi(H) < k$ for every proper subgraph $H$ of G, then G is color-critical or k-critical
Order of the largest clique

• Let $\alpha(G)$ denote the *independence number* of $G$, and $\omega(G)$ denote the order of the largest complete subgraph of $G$.

  – $\chi(G)$ may exceed $\omega(G)$. Consider $G = C_{2r+1} \vee K_s$
Cartesian Product

- The *Cartesian product* of graphs G and H, written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting $(u,v)$ adjacent to $(u',v')$ if and only if
  - (1) $u = u'$ and $vv' \in E(H)$, or
  - (2) $v = v'$ and $uu' \in E(G)$

- A graph G is *$m$-colorable* if and only if $G \square K_m$ has an independent set of size $n(G)$.

Also: $\chi(G \square H) = \max\{ \chi(G), \chi(H) \}$
Algorithm Greedy-Coloring

• The greedy coloring with respect to a vertex ordering \( v_1, \ldots, v_n \) of \( V(G) \) is obtained by coloring vertices in the order \( v_1, \ldots, v_n \), assigning to \( v_i \) the smallest indexed color not already used on its lower-indexed neighbors.
Results

• $\chi(G) \leq \Delta(G) + 1$

• If $G$ is an interval graph, then $\chi(G) = \omega(G)$

• If a graph $G$ has degree sequence $d_1 \geq \ldots \geq d_n$, then

\[
\chi(G) \leq 1 + \max_i \min\{ d_i, i-1 \}
\]
More results

- If $H$ is a $k$-critical graph, then $\delta(H) \geq k - 1$

- If $G$ is a graph, then $\chi(G) \leq 1 + \max_{H \subseteq G}\delta(H)$

- **Brooks Theorem:**

  If $G$ is a connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$. 
Mycielski’s Construction

- Builds from any given $k$-chromatic triangle-free graph $G$ a $k+1$-chromatic triangle-free super-graph $G'$.
  - Given $G$ with vertex set $V = \{v_1, \ldots, v_n\}$, add vertices $U = \{u_1, \ldots, u_n\}$ and one more vertex $w$.
  - Beginning with $G'[V] = G$, add edges to make $u_i$ adjacent to all of $N_G(v_i)$, and then make $N(w) = U$. Note that $U$ is an independent set in $G'$. 
Critical Graphs

• Suppose that $G$ is a graph with $\chi(G) > k$ and that $X, Y$ is a partition of $V(G)$. If $G[X]$ and $G[Y]$ are $k$-colorable, then the edge cut $[X,Y]$ has at least $k$ edges.

• [Dirac] Every $k$-critical graph is $k-1$ edge-connected.
Critical Graphs

Suppose $S$ is a set of vertices in a graph $G$. An $S$-component of $G$ is an induced sub-graph of $G$ whose vertex set consists of $S$ and the vertices of a component of $G - S$.

- If $G$ is $k$-critical, then $G$ has no cut set of vertices inducing a clique. In particular, if $G$ has a cut set, $S = \{ x, y \}$, then $x$ and $y$ are not adjacent and $G$ has an $S$-component $H$, such that $\chi(H + xy) \geq k$. 
Chromatic Recurrence

The function $\chi(G; k)$ counts the mappings $f: V(G) \rightarrow [k]$ that properly color $G$ from the set $[k] = \{1, \ldots, k\}$.

- In this definition, the $k$-colors need not all be used, and permuting the colors used produces a different coloring.

- If $G$ is a simple graph and $e \in E(G)$, then

$$\chi(G; k) = \chi(G - e; k) - \chi(G.e; k)$$
Line Graphs

The *line graph* of $G$, written $L(G)$, is a simple graph whose vertices are the edges of $G$, with $ef \in E(L(G))$ when $e$ and $f$ share a vertex of $G$.

- An Eulerian circuit in $G$ yields a spanning cycle in $L(G)$. The converse need not hold.
- A matching in $G$ is an independent set in $L(G)$; we have $\alpha'(G) = \alpha(L(G))$.
Edge Coloring

A \textit{k-edge-coloring} of G is a labeling \( f: E(G) \to [k] \)

- The labels are \textit{colors}
- The set of edges with one color is a \textit{color class}.
- A \textit{k-edge-coloring} is \textit{proper} if edges sharing a vertex have different colors; equivalently, each color class is a matching
- A graph is \textit{k-edge-colorable} if it has a proper \textit{k-edge-coloring}
- The \textit{edge-chromatic-number} \( \chi'(G) \) of a loop-less graph G is the least \( k \) such that G is \textit{k-edge-colorable}
Results

- \( \chi'(G) \geq \Delta(G) \).
- If \( G \) is a loop-less graph, then \( \chi'(G) \leq 2\Delta(G) - 1 \).
- If \( G \) is bipartite, then \( \chi'(G) = \Delta(G) \).

A regular graph \( G \) has a \( \Delta(G) \)-edge coloring if and only if it decomposes into 1-factors. We say that \( G \) is 1-factorable.

- Every simple graph with maximum degree \( \Delta \) has a proper \( \Delta + 1 \)-edge-coloring.